

An Example of Double Cross Coproducts with Non-trivial Left Coaction and Right Coaction in Strictly Braided Tensor Categories *

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Abstract

An example of double cross coproducts with both non-trivial left coaction and non-trivial right coaction in strictly braided tensor categories is given.

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0 Introduction and Preliminaries

The double cross coproducts in braided tensor categories have been studied by Y.Bespalov, B.Drabant and author in [2] [12]. However, hitherto any examples of double cross coproducts with both non-trivial left coaction and non-trivial right coaction in strictly braided tensor categories (i.e. the braiding is not symmetric) have not been found. Therefore Professor S.Majid asked if there is such example.

In this paper we first give the cofactorisation theorem of Hopf algebras in braided tensor categories. Using the cofactorisation theorem and Sweedler four dimensional Hopf algebra, we construct such example.

We denote the multiplication, comultiplication, evaluation d , coevaluation b , braiding and inverse braiding by



respectively. For convenience, we denote the inverse of morphism f by \bar{f} if f has an inverse.

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Since every braided tensor category is always equivalent to a strict braided tensor category by [12, Theorem 0.1], we can view every braided tensor category as a strict braided tensor category and use braiding diagrams freely.

1 The cofactorisation theorem of bialgebras in braided tensor categories

Throughout this section, we work in braided tensor category (\mathcal{C}, C) and assume that all Hopf algebras and bialgebras are living in (\mathcal{C}, C) unless otherwise stated. We give the cofactorisation theorem of bialgebras in braided tensor categories in this section.

We first recall the double bicrossproducts in [12]. Let H and A be two bialgebras in braided tensor categories and

$$\begin{aligned} \alpha : H \otimes A &\rightarrow A , \quad \beta : H \otimes A \rightarrow H , \\ \phi : A \rightarrow H \otimes A & , \quad \psi : H \rightarrow H \otimes A \end{aligned}$$

morphisms in \mathcal{C} .

$$\Delta_D =: \begin{array}{c} A \quad H \\ \text{---} \quad \text{---} \\ | \quad | \\ \phi \quad \psi \\ \text{---} \quad \text{---} \\ A \quad H \quad A \quad H \end{array} , \quad m_D =: \begin{array}{c} A \quad H \quad A \quad H \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \quad | \\ \alpha \quad \beta \\ \text{---} \quad \text{---} \\ A \quad H \end{array}$$

and $\epsilon_D = \epsilon_A \otimes \epsilon_H$, $\eta_D = \eta_A \otimes \eta_H$. We denote $(A \otimes H, m_D, \eta_D, \Delta_D, \epsilon_D)$ by

$$A_\alpha^\phi \bowtie_\beta^\psi H,$$

which is called the double bicrossproduct of A and H .

When ϕ and ψ are trivial, we denote $A_\alpha^\phi \bowtie_\beta^\psi H$ by $A_\alpha \bowtie_\beta H$. When α and β are trivial, we denote $A_\alpha^\phi \bowtie_\beta^\psi H$ by $A^\phi \bowtie^\psi H$. We call $A_\alpha \bowtie_\beta H$ a double cross product and denote it by $A \bowtie H$ in short. We call $A^\phi \bowtie^\psi H$ a double cross coproduct.

Theorem 1.1 (Factorisation theorem) (See [9, Theorem 7.2.3]) *Let X , A and H be bialgebras or Hopf algebras. Assume that j_A and j_H are bialgebra or Hopf algebra morphisms from A to X and H to X respectively. If $\xi =: m_X(j_A \otimes j_H)$ is an isomorphism from $A \otimes H$ onto X as objects in \mathcal{C} , then there exist morphisms*

$$\alpha : H \otimes A \rightarrow A \quad \text{and} \quad \beta : H \otimes A \rightarrow H$$

such that $A_\alpha \bowtie_\beta H$ becomes a bialgebra or Hopf algebra and ξ is a bialgebra or Hopf algebra isomorphism from $A_\alpha \bowtie_\beta H$ onto X .

Proof. Set

$$\zeta := \begin{array}{c} H \\ \text{\scriptsize } j_H \\ \text{\scriptsize } j_A \\ \text{\scriptsize } \xi \\ A \quad H \end{array}, \quad \alpha := \begin{array}{c} H \\ \text{\scriptsize } \zeta \\ A \\ A \end{array} \quad \text{and} \quad \beta := \begin{array}{c} H \\ \text{\scriptsize } \zeta \\ A \\ \text{\scriptsize } \epsilon \\ H \end{array}.$$

We see

Thus

$$\begin{array}{c} H \quad H \quad A \\ | \quad | \quad | \\ \textcircled{\zeta} \quad \textcircled{\zeta} \end{array} = \begin{array}{c} H \quad H \quad A \\ | \quad | \quad | \\ A \quad H \quad \textcircled{\zeta} \end{array} \dots \dots \quad (1)$$

Similarly we have

$$\begin{array}{c} H \quad A \quad A \\ | \quad | \quad | \\ \textcircled{\sigma} \quad \textcircled{\sigma} \quad \textcircled{\sigma} \\ | \quad | \quad | \\ A \quad H \end{array} = \begin{array}{c} H \quad A \quad A \\ | \quad | \quad | \\ \textcircled{\sigma} \quad \textcircled{\sigma} \quad \textcircled{\sigma} \\ | \quad | \quad | \\ A \quad H \end{array} . \quad \dots \dots (2)$$

We also have

$$\zeta(\eta \otimes id) = id \otimes \eta \quad \text{and} \quad \zeta(id \otimes \eta) = \eta \otimes id . \quad \dots \dots \quad (3)$$

It is clear that ζ is a coalgebra morphism from $H \otimes A$ to $A \otimes H$, since j_A , j_H and m_X all are coalgebra homomorphisms. Thus we have

$$\begin{array}{c} H \\ | \\ \text{---} \\ | \\ A \end{array}
 \quad
 \begin{array}{c} A \\ | \\ \text{---} \\ | \\ H \end{array}
 \quad
 =
 \quad
 \begin{array}{c} H \\ | \\ \text{---} \\ | \\ \zeta \end{array}
 \quad
 \text{and } (\epsilon \otimes \epsilon)\zeta = (\epsilon \otimes \epsilon). \quad \dots \dots (4)$$

We now show that (A, α) is an H -module coalgebra:

$$\begin{array}{c}
 H \quad HA \\
 \diagdown \quad \diagup \\
 \text{---} \quad \text{---} \\
 \alpha \quad \epsilon \\
 \text{---} \quad \text{---} \\
 A \quad \epsilon
 \end{array}
 \quad = \quad
 \begin{array}{c}
 H \quad H \quad A \\
 \diagdown \quad \diagup \quad \text{---} \\
 \text{---} \quad \text{---} \quad \text{---} \\
 \text{---} \quad \text{---} \quad \text{---} \\
 \zeta \quad \epsilon \\
 \text{---} \quad \text{---} \\
 A \quad \epsilon
 \end{array}
 \quad \text{by } (1) \quad
 \begin{array}{c}
 H \quad H \quad A \\
 \diagdown \quad \diagup \quad \text{---} \\
 \text{---} \quad \text{---} \quad \text{---} \\
 \zeta \quad \zeta \\
 \text{---} \quad \text{---} \\
 A \quad \epsilon \\
 \text{---} \quad \text{---} \\
 A \quad \epsilon
 \end{array}
 \quad = \quad
 \begin{array}{c}
 HH \quad A \\
 \diagdown \quad \diagup \\
 \text{---} \quad \text{---} \\
 \alpha \quad \alpha \\
 \text{---} \quad \text{---} \\
 A \quad \epsilon
 \end{array}$$

and $\alpha(\eta \otimes id_A) = (id_A \otimes \epsilon)\zeta(\eta \otimes id_A)$ by (3) $= id_A$.

We see that $\epsilon \circ \alpha = (\epsilon \otimes \epsilon) \zeta$ by (4) $\epsilon \otimes \epsilon$ and

Thus (A, α) is an H -module coalgebra. Similarly, we can show that (H, β) is an A -module coalgebra.

Now we show that conditions $(M1)$ – $(M4)$ in [12,p37] hold. By (3), we easily know that $(M1)$ holds. Next we show that $(M2)$ holds.

$$\begin{array}{c}
 \text{Diagram 1:} \\
 \begin{array}{c}
 \text{Left: } H \xrightarrow{\alpha} \text{ and } A \xrightarrow{\beta} A. \\
 \text{Middle: } H \xrightarrow{\alpha} \text{ and } A \xrightarrow{\zeta} A. \\
 \text{Right: } H \xrightarrow{\zeta} A \xrightarrow{\zeta} A. \\
 \text{Equation: } \text{Left} = \text{Middle} \xrightarrow{(4)} \text{Right}.
 \end{array}
 \end{array}$$

Thus (M2) holds. Similarly, we can get the proofs of (M3) and (M4). Consequently, $A_\alpha \bowtie_\beta H$ is a bialgebra or Hopf algebra by [12, Corollary 1.8]. It suffices to show that ζ is a bialgebra

morphism from $A_\alpha \bowtie_\beta H$ to X . Let $D = A_\alpha \bowtie_\beta H$. Since

$$\begin{array}{c}
 \text{Diagram showing } H \text{ and } A \text{ as bialgebras with comultiplications } \Delta_H \text{ and } \Delta_A, \text{ and counits } \eta_A \text{ and } \eta_H. \\
 \text{Diagram by (4) shows } \xi \text{ as a morphism from } A_\alpha \bowtie_\beta H \text{ to } X. \\
 \text{Resulting diagram shows } D \text{ as a bialgebra with comultiplication } \Delta_D \text{ and counit } \eta_D.
 \end{array}$$

we have that ξ is a bialgebra morphism from $A_\alpha \bowtie_\beta H$ to X by [12, Lemma 2.5]. \square

Theorem 1.2 (Co-factorisation theorem) *Let X , A and H be bialgebras or Hopf algebras. Assume that p_A and p_H are bialgebra or Hopf algebra morphisms from X to A and X to H , respectively. If $\xi = (p_A \otimes p_H)\Delta_X$ is an isomorphism from X onto $A \otimes H$ as objects in \mathcal{C} , then there exist morphisms:*

$$\phi : A \rightarrow H \otimes A \quad \text{and} \quad \psi : H \rightarrow H \otimes A$$

such that $A^\phi \bowtie^\psi H$ becomes a bialgebra or Hopf algebra and ξ is a bialgebra or Hopf algebra isomorphism from X to $A^\phi \bowtie^\psi H$.

Proof. Set

$$\begin{array}{ccc}
 \text{Diagram showing } A \text{ and } H \text{ as bialgebras with comultiplications } \Delta_A \text{ and } \Delta_H, \text{ and counits } \eta_H \text{ and } \eta_A. \\
 \text{Diagram shows } \xi \text{ as a morphism from } A \otimes H \text{ to } H \otimes A. \\
 \text{Resulting diagram shows } \phi \text{ as a morphism from } A \text{ to } H \otimes A \text{ and } \psi \text{ as a morphism from } H \text{ to } H \otimes A.
 \end{array}$$

We can complete the proof by turning upside down the diagrams in the proof of the preceding theorem. \square

From now on, we always consider Hopf algebras over field k and the diagram

$$\begin{array}{c}
 U \quad V \\
 \diagtimes \\
 V \quad U
 \end{array}$$

always denotes the ordinary twisted map: $x \otimes y \longrightarrow y \otimes x$. Our diagrams only denote homomorphisms between vector spaces, so two diagrams can have the additive operation.

Let H be an ordinary bialgebra and (H_1, R) an ordinary quasitriangular Hopf algebra over field k . Let f be a bialgebra homomorphism from H_1 to H . Then there exists a bialgebra B , written as $B(H_1, f, H)$, living in $(_{H_1}\mathcal{M}, C^R)$. Here $B(H_1, f, H) = H$ as algebra, its counit is ϵ_H ,

and its comultiplication and antipode are

$$\begin{array}{c}
 \text{Diagram showing } \Delta_B : B \rightarrow H \otimes H \\
 \text{Diagram showing } S_B : B \rightarrow B
 \end{array}$$

The left diagram shows the comultiplication Δ_B mapping B to $H \otimes H$. It consists of two parallel strands labeled H and H merging at the bottom into a single strand labeled B . This is equated to a more complex diagram where B splits into two strands labeled f and fS , which then merge into a single strand labeled ad , which finally merges with the original H strands. The top part of this complex diagram has a node labeled Δ_H and a node labeled R .

The right diagram shows the antipode S_B mapping B to B . It consists of a vertical strand labeled B that splits into two strands labeled f and f , which then merge into a single strand labeled ad , which finally merges with the original B strand. The top part of this diagram has a node labeled S_B and a node labeled S .

(see [8, Theorem 4.2]), respectively. In particular, when $H = H_1$ and $f = id_H$, $B(H_1, f, H)$ is a braided group, called the braided group analogue of H and written as \underline{H} .

R is called a weak R -matrix of $A \otimes H$ if R is invertible under convolution with

$$\begin{array}{c}
 \text{Diagram showing } R \text{ is a weak } R\text{-matrix} \\
 \text{Diagram showing } R \text{ is a weak } R\text{-matrix}
 \end{array}$$

The left diagram shows the convolution of R with A and AH . It consists of two parallel strands labeled A and AH merging at the bottom into a single strand labeled A . This is equated to a more complex diagram where A and AH split into two strands each, labeled R , which then merge into a single strand labeled H . The right diagram shows the convolution of R with AH and H . It consists of two parallel strands labeled AH and H merging at the bottom into a single strand labeled A . This is equated to a more complex diagram where AH and H split into two strands each, labeled R , which then merge into a single strand labeled HH .

Let (A, P) and (H, Q) be ordinary finite-dimensional quasitriangular Hopf algebras over field k . Let R be a weak R -matrix of $A \otimes H$. For any $U, V \in CW(A \otimes H) =: \{U \in A \otimes H \mid U \text{ is a weak } R\text{-matrix and in the center of } A \otimes H\}$,

$$R_D =: \sum R' P' U' \otimes Q'(R^{-1})'' V'' \otimes P''(R^{-1})' V' \otimes R'' Q'' U''$$

is a quasitriangular structure of D and every quasitriangular structure of D is of this form ([3, Theorem 2.9]), where $R = \sum R' \otimes R''$, etc.

Lemma 1.3 *Under the above discussion, then*

- (i) $\pi_A : D \rightarrow A$ and $\pi_H : D \rightarrow H$ are bialgebra or Hopf algebra homomorphisms, respectively. Here π_A and π_H are trivial action, that is, $\pi_A(h \otimes a) = \epsilon(h)a$ for any $a \in A, h \in H$.
- (ii) $B(D, \pi_A, A) = \underline{A}$ and $B(D, \pi_H, H) = \underline{H}$.
- (iii) $\pi_{\underline{A}} : \underline{D} \rightarrow \underline{A}$ and $\pi_{\underline{H}} : \underline{D} \rightarrow \underline{H}$ are bialgebra or Hopf algebra homomorphisms, respectively.

Proof. (i) It is clear.

(ii) It is enough to show $\Delta_B = \Delta_{\underline{A}}$ since $B = \underline{A}$ as algebras, where $B =: B(D, \pi_A, A)$. See

$$\begin{array}{c}
A \\
\Delta_A \\
\downarrow \\
B \quad B \\
\Delta_B \\
\downarrow \\
A \quad A
\end{array} =
\begin{array}{c}
A \\
\Delta_A \\
\downarrow \\
B \quad B \\
\Delta_B \\
\downarrow \\
A \quad A
\end{array} =
\begin{array}{c}
A \\
\Delta_A \\
\downarrow \\
B \quad B \\
\Delta_B \\
\downarrow \\
A \quad A
\end{array} =
\begin{array}{c}
A \\
\Delta_A \\
\downarrow \\
\underline{A} \quad \underline{A}
\end{array}.$$

Thus $\Delta_B = \Delta_A$. Similarly, we have $B(D, \pi_H, H) = \underline{H}$.

(iii) See

$$\begin{array}{c}
D \\
\Delta_D \\
\downarrow \\
\underline{H} \quad \underline{H} \\
\Delta_{\underline{H}} \\
\downarrow \\
H \quad H
\end{array} =
\begin{array}{c}
D \\
\Delta_D \\
\downarrow \\
\underline{H} \quad \underline{H} \\
\Delta_{\underline{H}} \\
\downarrow \\
H \quad H
\end{array} =
\begin{array}{c}
D \\
\Delta_D \\
\downarrow \\
\underline{H} \quad \underline{H} \\
\Delta_{\underline{H}} \\
\downarrow \\
H \quad H
\end{array} \text{ by (i)} =
\begin{array}{c}
D \\
\Delta_D \\
\downarrow \\
\underline{H} \quad \underline{H} \\
\Delta_{\underline{H}} \\
\downarrow \\
H \quad H
\end{array} =
\begin{array}{c}
A \\
\epsilon \\
\downarrow \\
\underline{H} \quad \underline{H} \\
\Delta_{\underline{H}} \\
\downarrow \\
H \quad H
\end{array} =
\begin{array}{c}
H \\
\Delta_D \\
\downarrow \\
\underline{H} \quad \underline{H} \\
\Delta_{\underline{H}} \\
\downarrow \\
H \quad H
\end{array}$$

and $\epsilon \circ \pi_{\underline{H}} = \epsilon$. Thus $\pi_{\underline{H}}$ is a coalgebra homomorphism.

Since the multiplications in \underline{D} and \underline{H} are the same as in D and H , respectively, we have that $\pi_{\underline{H}}$ is an algebra homomorphism by (i). Similarly, we can show that $\pi_{\underline{A}}$ is a bialgebra homomorphism. \square

We now investigate the relation among braided group analogues of quasitriangular Hopf algebras A and H and their double cross coproduct $D = A \bowtie^R H$.

Theorem 1.4 *Under the above discussion, let $\xi = (\pi_{\underline{A}} \otimes \pi_{\underline{H}})\Delta_{\underline{D}}$. Then*

$$\xi =
\begin{array}{c}
A \quad H \\
\downarrow \quad \downarrow \\
A \quad H \\
\Delta_{\underline{D}} \\
\downarrow \\
S \quad ad \\
\downarrow \\
A \quad H
\end{array}$$

and ξ is surjective, where ad denotes the left adjoint action of H .

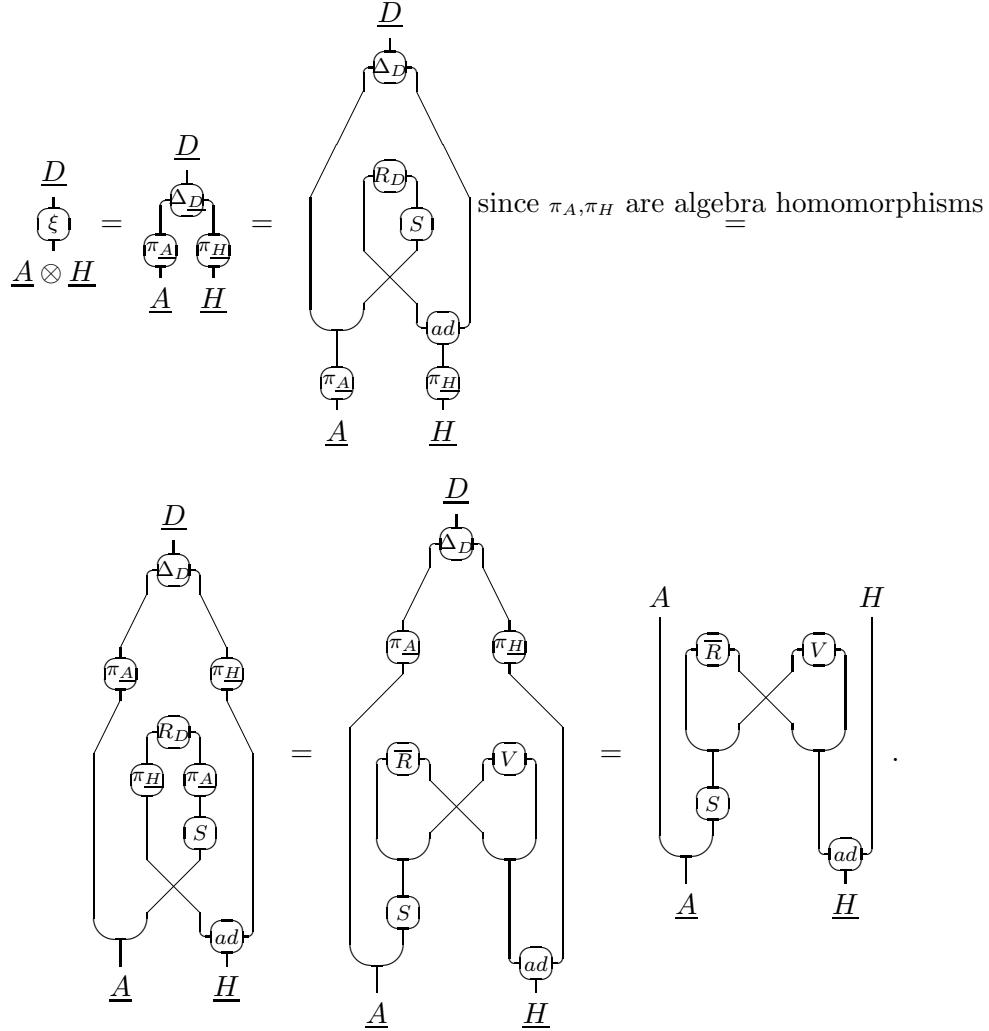
(ii) Furthermore, if A and H are finite-dimensional, then ξ is a bijective map from \underline{D} onto $\underline{A} \otimes \underline{H}$. That is, in braided tensor category $({}_D\mathcal{M}, C^{RD})$, there exist morphisms ϕ and ψ such that

$$\underline{D} \cong \underline{A}^\phi \bowtie^\psi \underline{H} \quad (\text{as Hopf algebras})$$

and the isomorphism is $(\pi_{\underline{A}} \otimes \pi_{\underline{H}})\Delta_{\underline{D}}$.

(iii) If H is commutative or $V = R$, then $\xi = id_{\underline{D}}$.

Proof. (i)



(ii) By the proof of (i), ξ is bijective and

Applying the cofactorization theorem 1.2, we complete the proof of (ii).

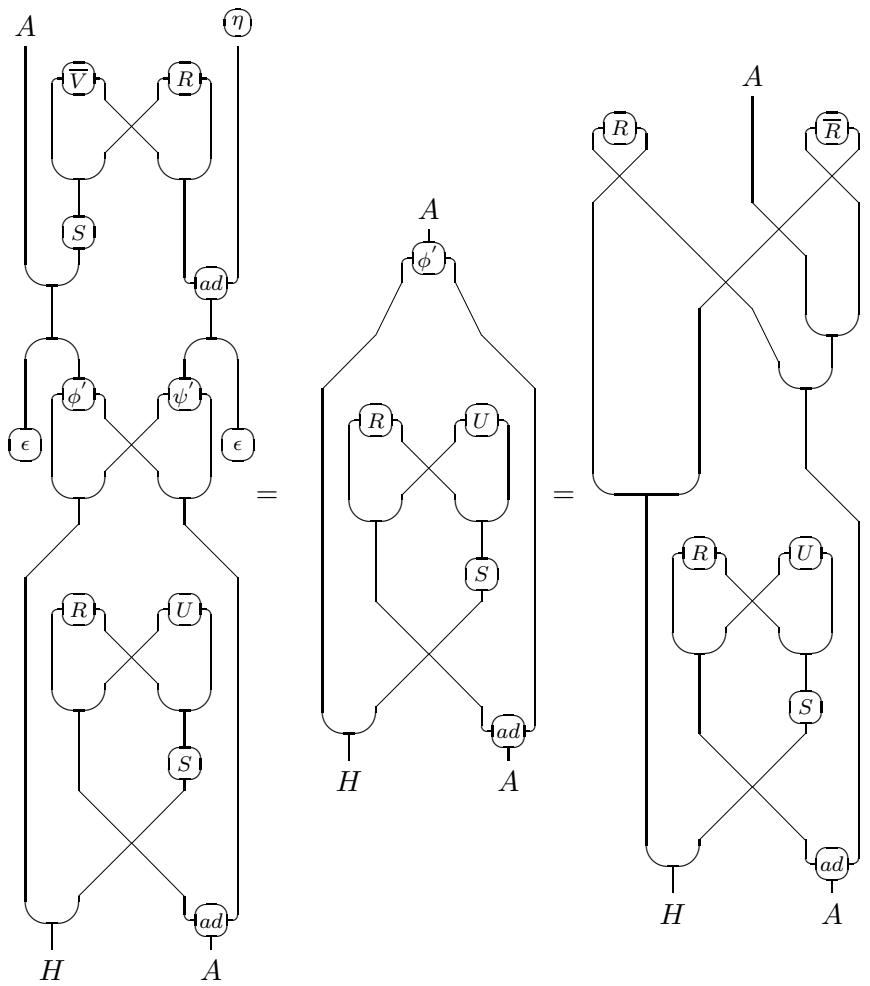
(iii) follows from (i). \square

Remark. Under the assumption of Theorem 1.4, if we set

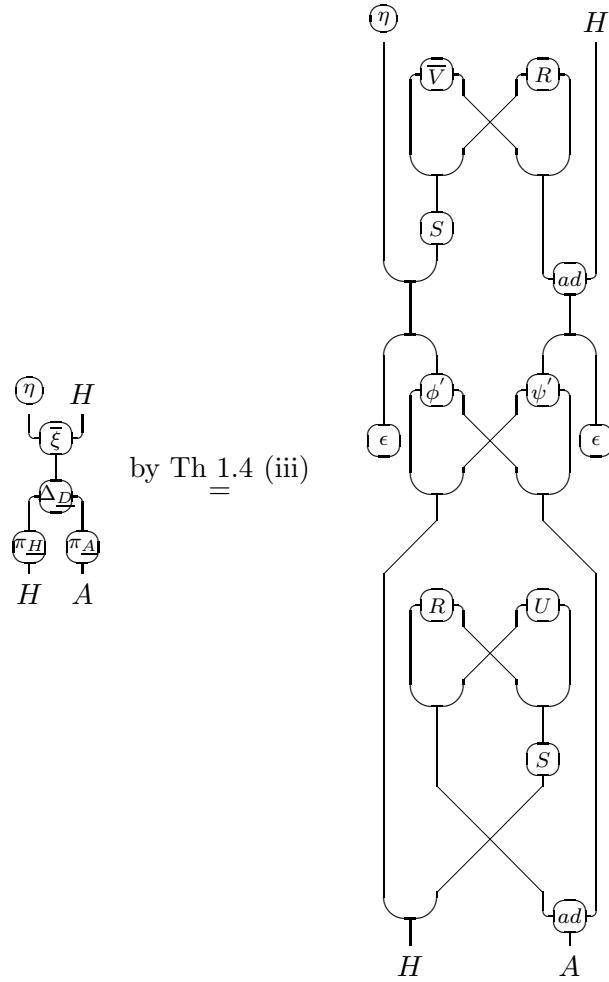
$$\begin{array}{c}
 \text{Diagram 1: } A \xrightarrow{\phi} H \text{ and } A \xrightarrow{\bar{R}} H \\
 \text{Diagram 2: } H \xrightarrow{\psi'} A \text{ and } H \xrightarrow{\bar{R}} A
 \end{array}$$

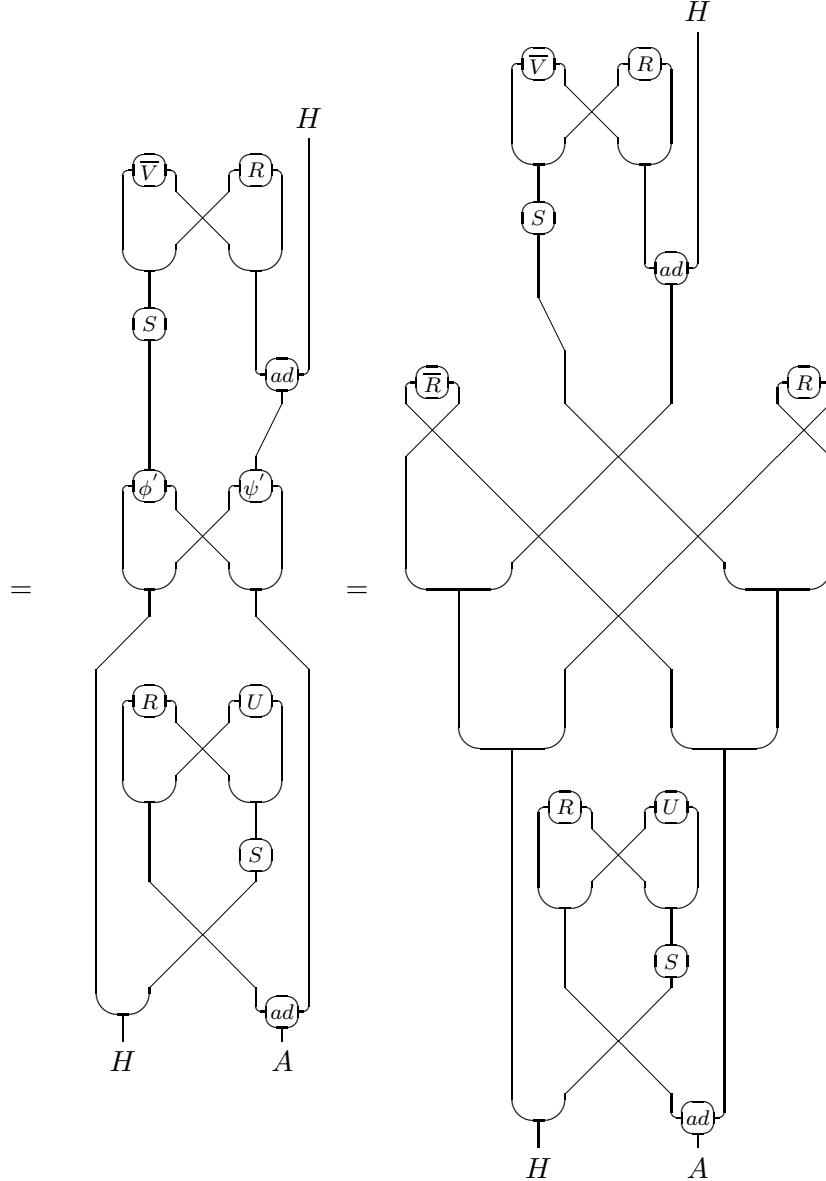
then $A^\phi \bowtie^{\psi'} H = A \bowtie^R H$ by [3, Lemma 1.3]. We now see the relation among ϕ, ψ, R, ϕ' and ψ' :

$$\phi \stackrel{\text{by proof of Th 1.2}}{=} \zeta(id \otimes \eta) = \begin{array}{c} A \\ \uparrow \bar{\xi} \\ \Delta_D \\ \pi_H \quad \pi_A \\ H \quad A \end{array} \stackrel{\text{by Th 1.4 (iii)}}{=}$$



and ψ by proof of Th 1.2 $\zeta(\eta \otimes id) =$





Furthermore, $\psi = \psi'$ when H is commutative or $R = V$. In this case, $\underline{A^{\phi'}} \bowtie^{\psi'} \underline{H} = \underline{A^{\phi'} \bowtie^{\psi'} H} = \underline{A \bowtie^R H}$ as Hopf algebras living in braided tensor category $(D\mathcal{M}, C^{RD})$.

2 An example

In this section, using preceding cofactorisation theorem, we give an example of double cross coproducts with both non-trivial left coaction and non-trivial right coaction in strictly braided tensor categories.

Let $H^* = Hom(H, k)$ be the dual of finite-dimensional Hopf algebra H . H^* can become a Hopf algebra under convolution ([10, Theorem 9.1.3]). That is, for any $f, g \in H^*, h, h' \in H$,

$$(f * g)(h) = \sum_{(h)} f(h_1)g(h_2), \Delta_{H^*}(f)(h \otimes h') = f(hh'), S_{H^*}(f)(h) = f(S(h)).$$

Assume $\{e_{x_i} \mid i = 1, 2, \dots, n\}$ is the dual basis of $\{x_i \mid i = 1, 2, \dots, n\}$. Define $d_H = \begin{cases} H^* \otimes H \rightarrow k \\ f \otimes h \rightarrow f(h) \end{cases}$ and $b_H = \begin{cases} k \rightarrow H \otimes H^* \\ 1 \rightarrow \sum_{i=1}^n x_i \otimes e_{x_i} \end{cases}$. d_H and b_H are called evaluation and coevaluation of H , respectively. It is clear that

$$\begin{array}{ccc}
\text{Diagram: } & & \\
\begin{array}{c} H^* \\ \text{---} \\ \text{U-shaped line} \\ \text{---} \\ H^* \end{array} & = & \begin{array}{c} H^* \\ | \\ H^* \end{array}, \\
& & \\
\begin{array}{c} H \\ \text{---} \\ \text{U-shaped line} \\ \text{---} \\ H \end{array} & = & \begin{array}{c} H \\ | \\ H \end{array}, \\
& & \\
\begin{array}{c} H^* \quad H^* \\ \text{---} \quad \text{---} \\ \text{U-shaped line} \quad \text{---} \\ \text{---} \quad \text{---} \\ H^* \end{array} & = & \begin{array}{c} H^* \quad H^* \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ H^* \end{array} \text{ and } \begin{array}{c} H^* \\ \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ H^* \quad H^* \end{array} = \begin{array}{c} H^* \\ \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ H^* \quad H^* \end{array}.
\end{array}$$

Note the multiplication and comultiplication of H^* exactly are anti-multiplication and anti-comultiplication in [8, Proposition 2.4].

$$\text{Let } A = H^{*cop}, \quad \tau = \begin{array}{c} H \quad A \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \end{array} \quad \text{and} \quad [b] = \begin{array}{c} \eta \quad \eta \\ \text{---} \quad \text{---} \\ AH \quad AH \end{array}.$$

Thus τ is a skew pairing and the Drinfeld Double $D(H) = A \bowtie_\tau H$ (see [4] [5]). Furthermore, $[b]$ is a quasitriangular structure of $D(H)$ by [10, Theorem 10.3.6].

Lemma 2.1 *Let H be a finite-dimensional Hopf algebra with $\dim H > 1$. Then Drinfeld double $(D(H), [b])$ is not triangular.*

Proof. Assume that x'_i 's are a basis of H with $x_1 = 1_H$. Set $x =: x_2$ and see

$$\begin{array}{c}
x \otimes \epsilon \quad \epsilon \otimes e_x = \begin{array}{c} \eta \quad \eta \\ \text{---} \quad \text{---} \\ x \quad \epsilon \quad \epsilon \quad e_x \end{array} = 0 \quad \text{and} \\
x \otimes \epsilon \quad \epsilon \otimes e_x = \begin{array}{c} \eta \quad \eta \\ \text{---} \quad \text{---} \\ x \quad \epsilon \quad \epsilon \quad e_x \end{array} = 1. \quad \text{Thus} \quad \begin{array}{c} \text{---} \quad \text{---} \\ D \quad D \end{array} \neq \begin{array}{c} [b] \\ \text{---} \\ D \quad D \end{array},
\end{array}$$

which implies that $[b]$ is not triangular. \square

Let us recall Sweedler's four dimensional Hopf algebra H_4 . That is, H_4 is a Hopf algebra generated g and x with relations

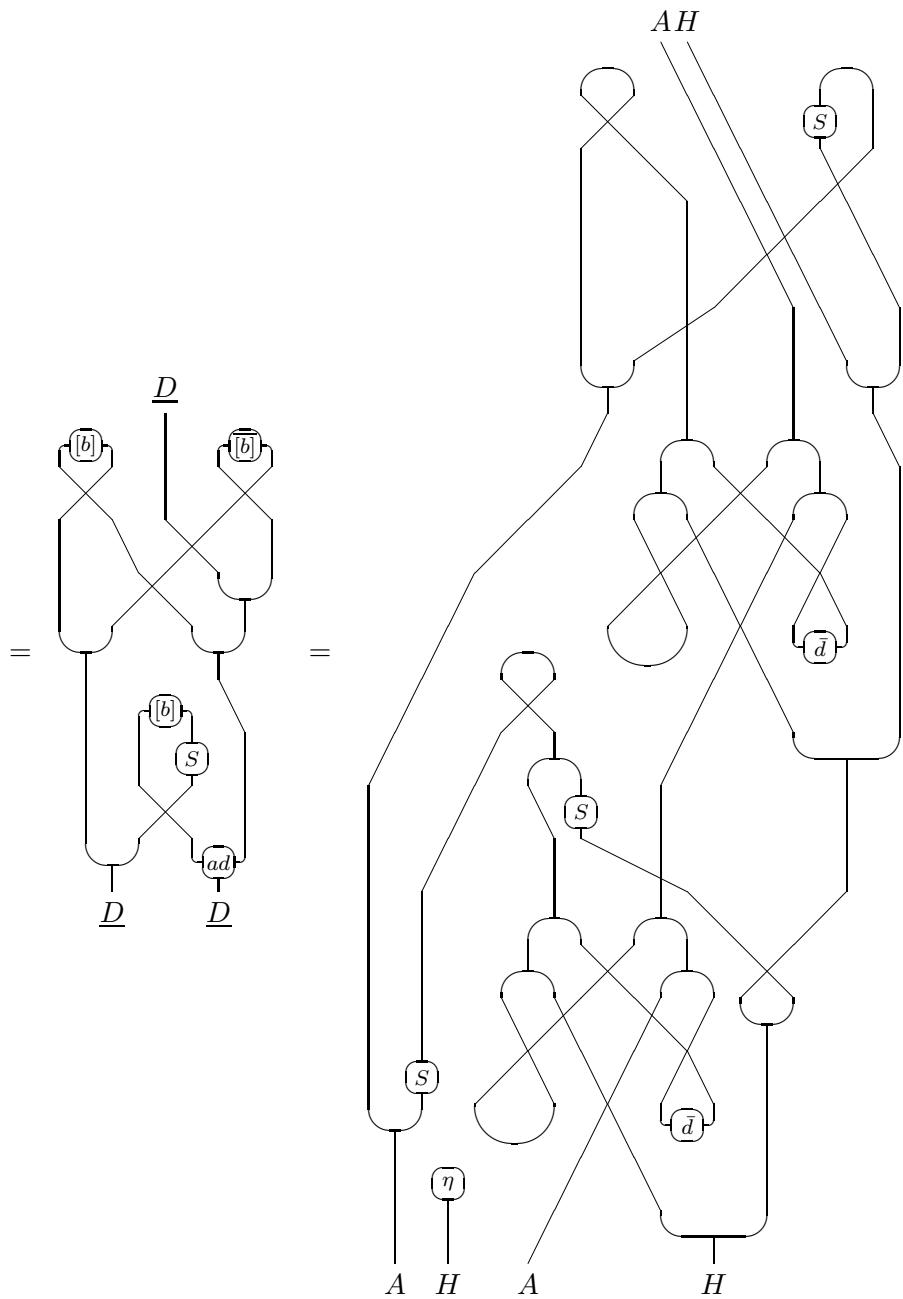
$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx$$

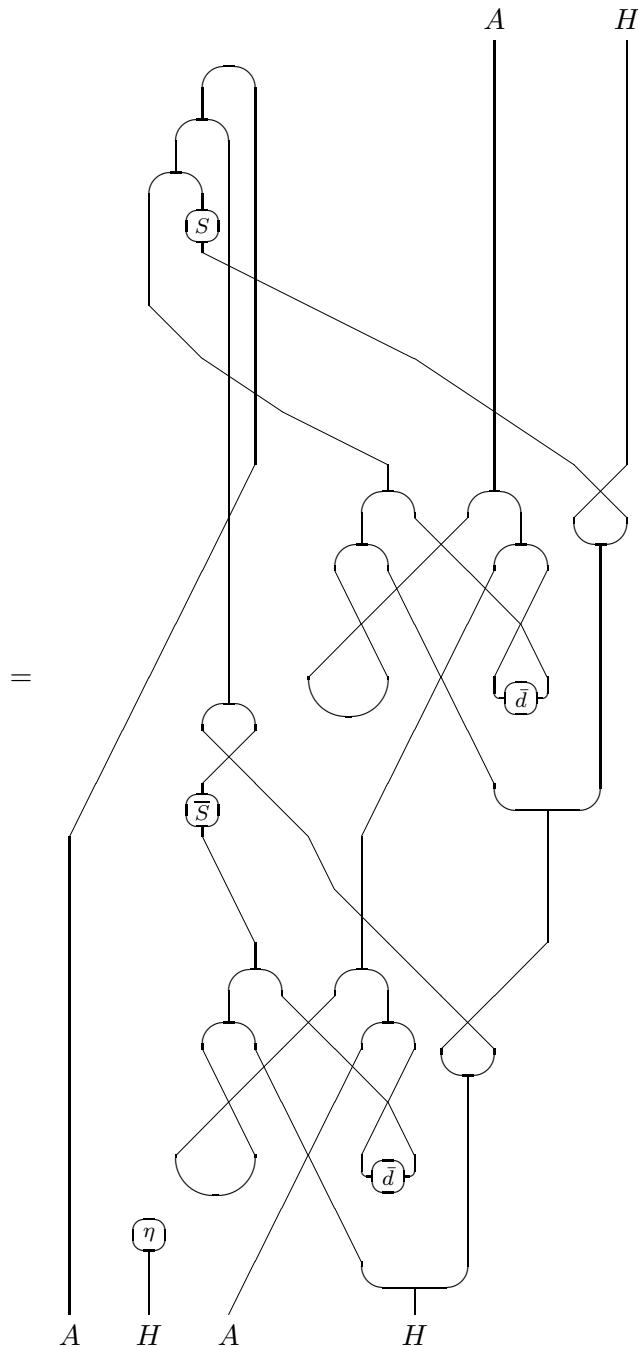
and $\Delta(x) = x \otimes 1 + g \otimes x$, $\Delta(g) = g \otimes g$, $\epsilon(x) = 0$, $\epsilon(g) = 1$, $S(x) = xg$, $S(g) = g$. Let $\{e_1, e_g, e_x, e_{gx}\}$ denote the dual basis of $\{1, g, x, gx\}$.

Example 2.2 Let H be Sweedler's four dimensional Hopf algebra over field k with $\text{char } k \neq 2$. Let $D = D(H)$. Thus $B =: D \bowtie^{[b]} D$ is quasitriangular, but it is not triangular by Lemma 2.1. Considering [3, Theorem 2.5], B has a quasitriangular structure R_B , defined in preceding Theorem 1.4 with $U = V = 1 \otimes 1$, and R_B never is triangular. Thus $(B\mathcal{M}, C^{R_B})$ is a strictly braided tensor category by [10, Theorem 10.4.2 (3)]. It follows from Theorem 1.4 (ii) that $D \bowtie^{[b]} D \cong D^\phi \bowtie^\psi D$ for some ϕ and ψ . Furthermore, $D^\phi \bowtie^\psi D$ is a double cross coproduct. We shall show that both left coaction ϕ and right coaction ψ are non-trivial.

Proof.

$$\begin{array}{c} \underline{D} \\ \downarrow \\ \text{---} \circlearrowleft \text{---} \\ \underline{D} \quad \underline{D} \end{array}$$





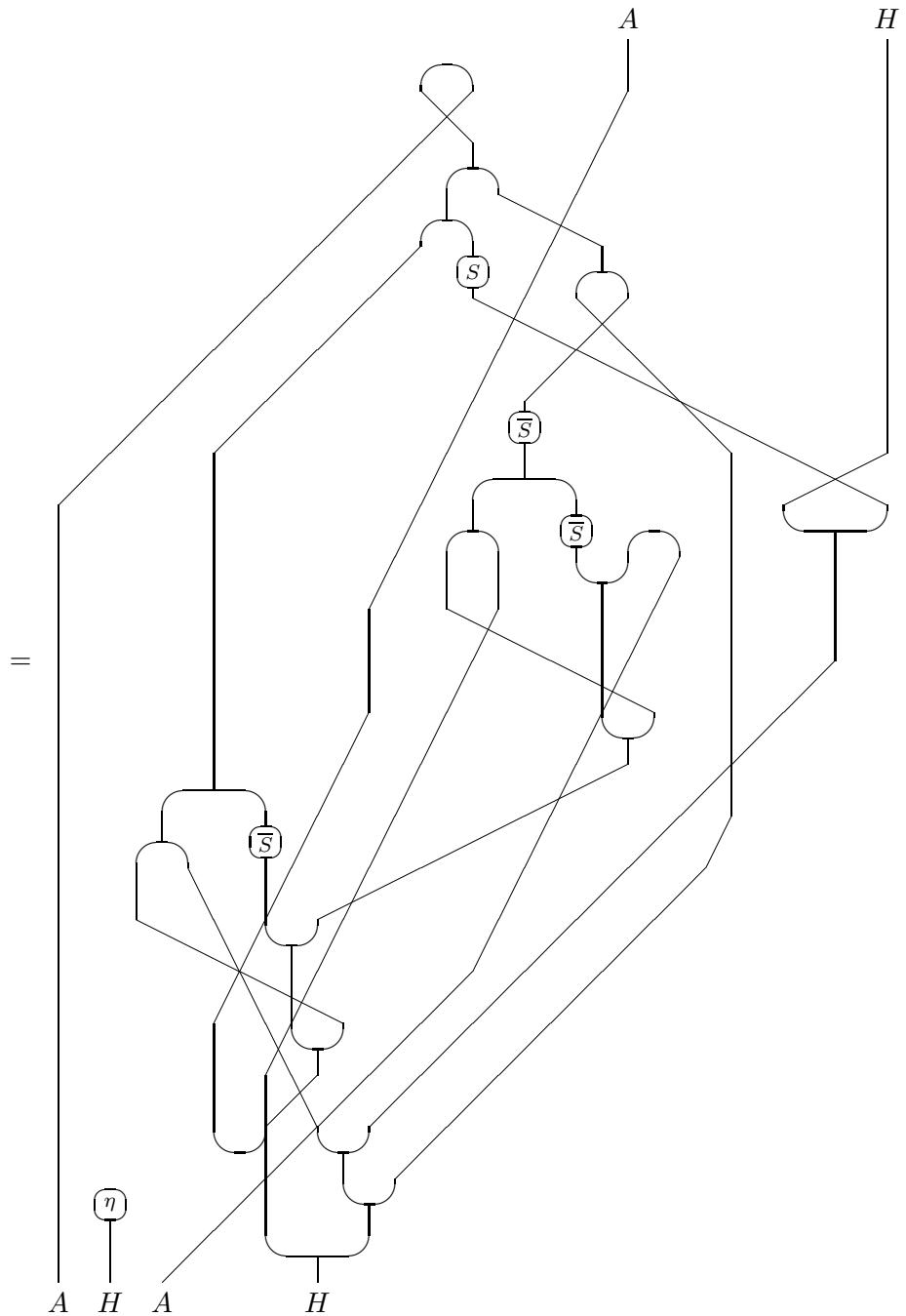
Compute:

$$\begin{array}{c}
 A = \text{Diagram} \\
 \text{and } \lambda =: \text{Diagram} \\
 H \quad A \\
 = \text{Diagram} = \text{Diagram}
 \end{array}$$

The diagram for A shows a vertical line labeled A with two loops at the top and bottom. The diagram for λ shows two vertical lines with loops at the top and bottom, with a small circle containing d at the bottom right. The equation $H = A$ is shown with two string diagrams: one on the left where H has a loop at the bottom and A has a loop at the top; and one on the right where H has a loop at the top and A has a loop at the bottom. Both diagrams have a small circle containing d at the bottom right.

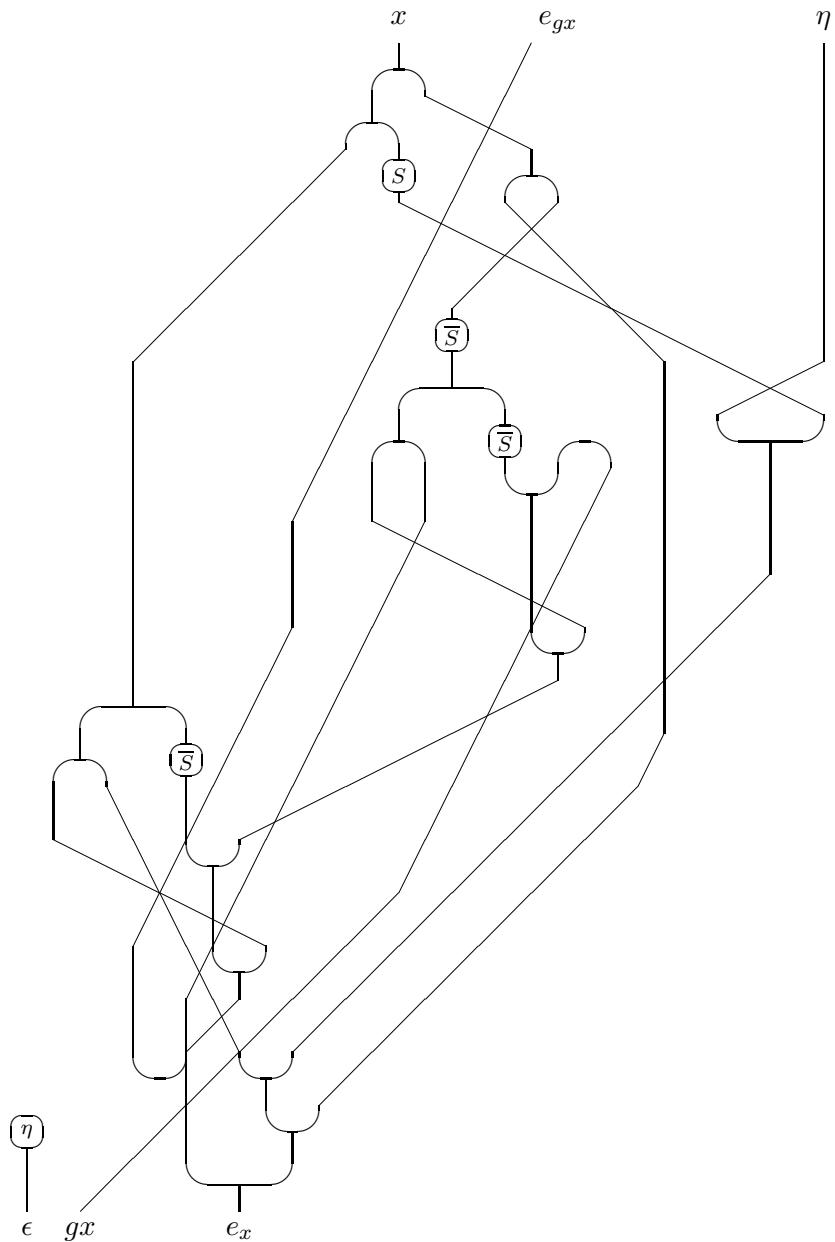
Also,

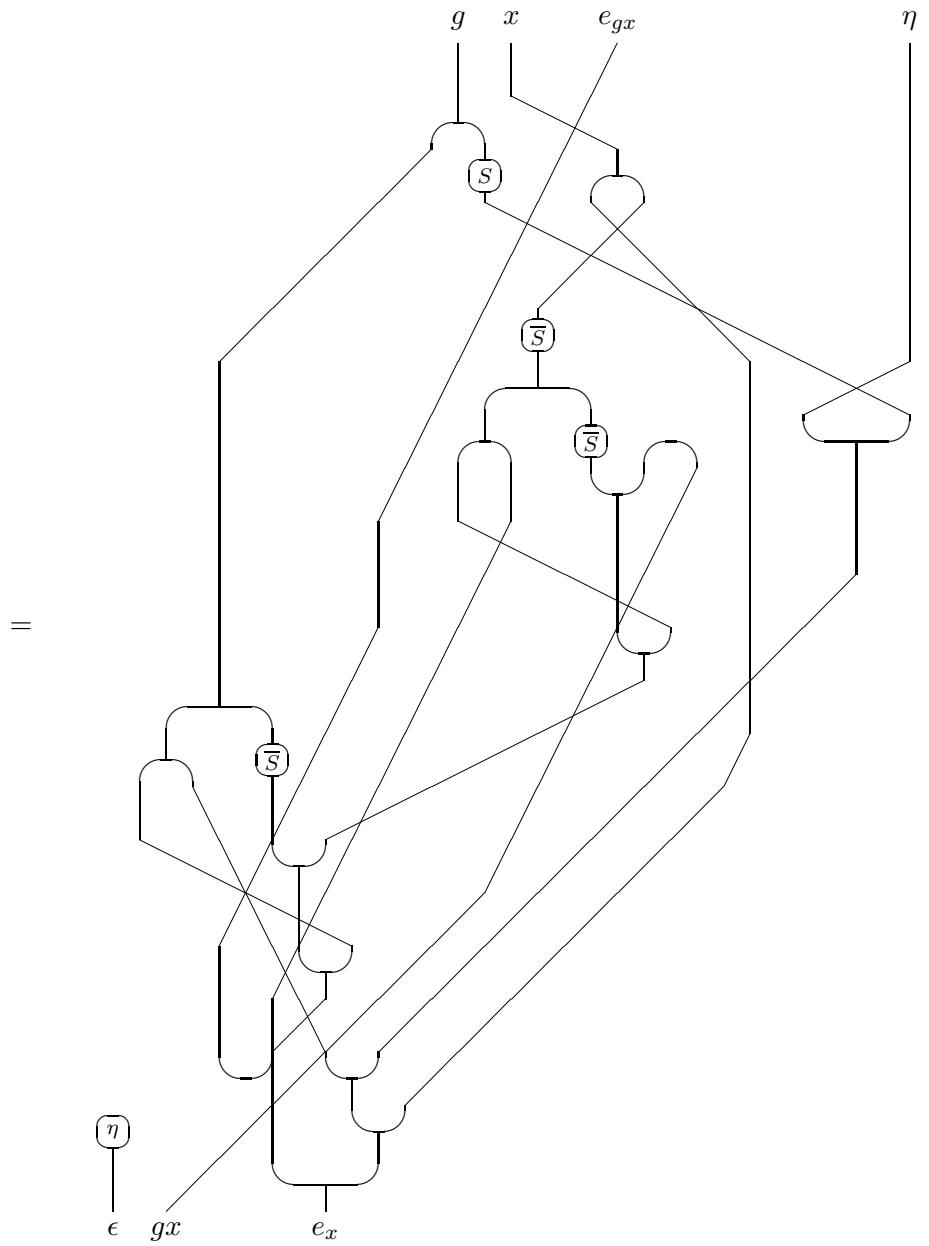
$$\frac{D}{\underline{D} \quad \underline{D}} \circ \phi$$

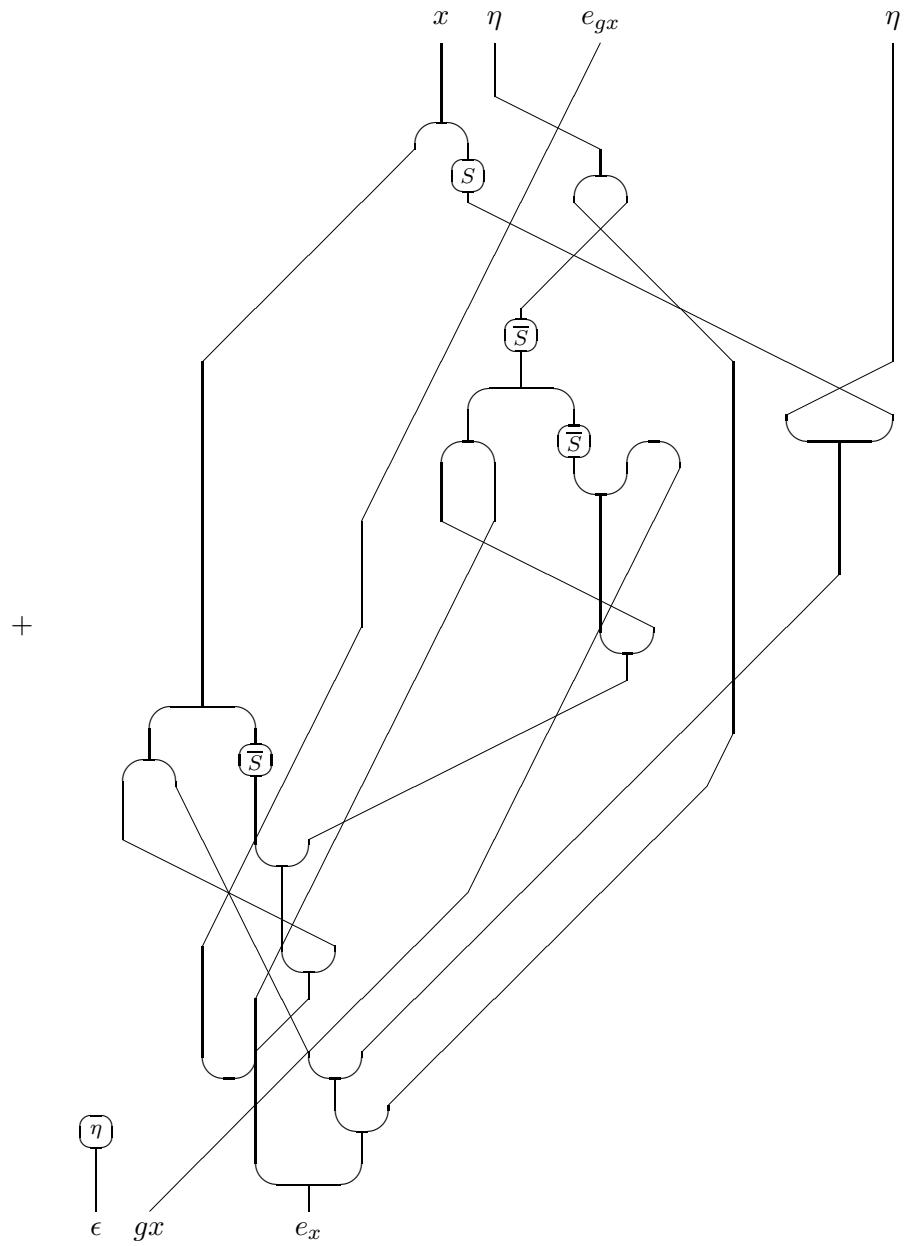


and

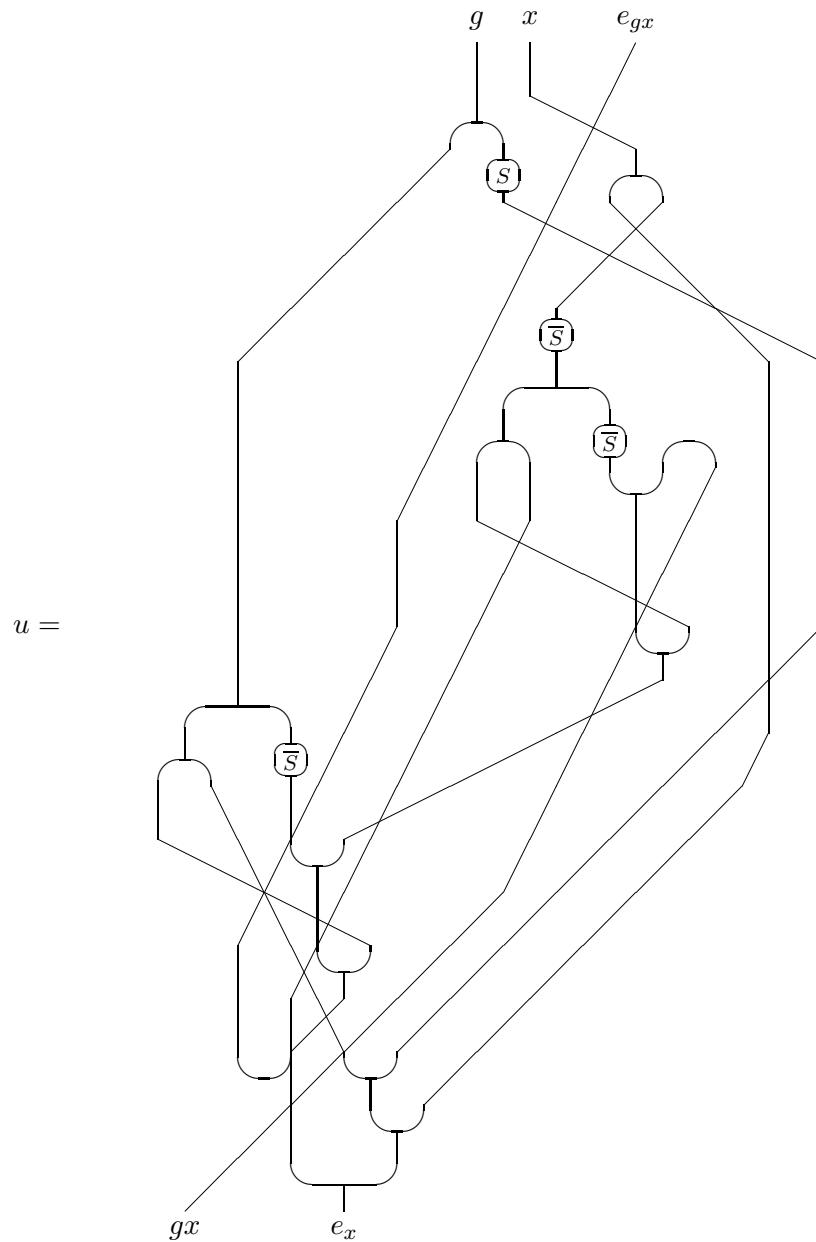
$$\begin{array}{c}
 e_{gx} \otimes \eta \\
 \downarrow \phi \\
 x \otimes \epsilon gx \otimes e_x
 \end{array}
 =$$

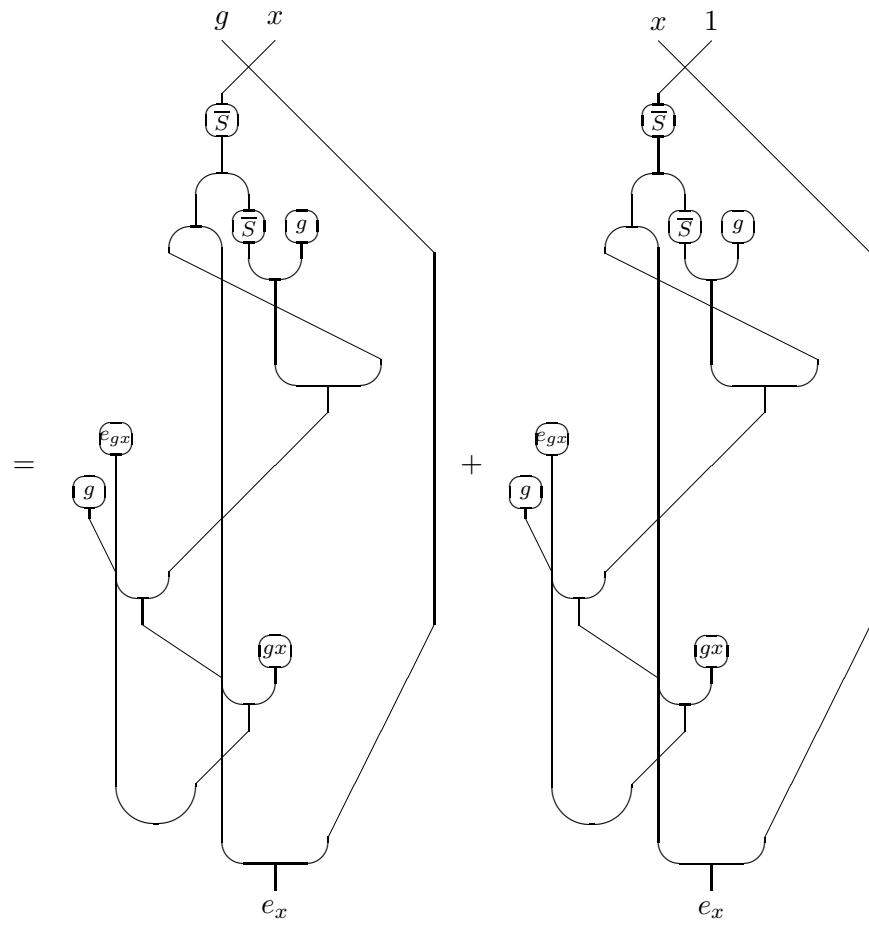


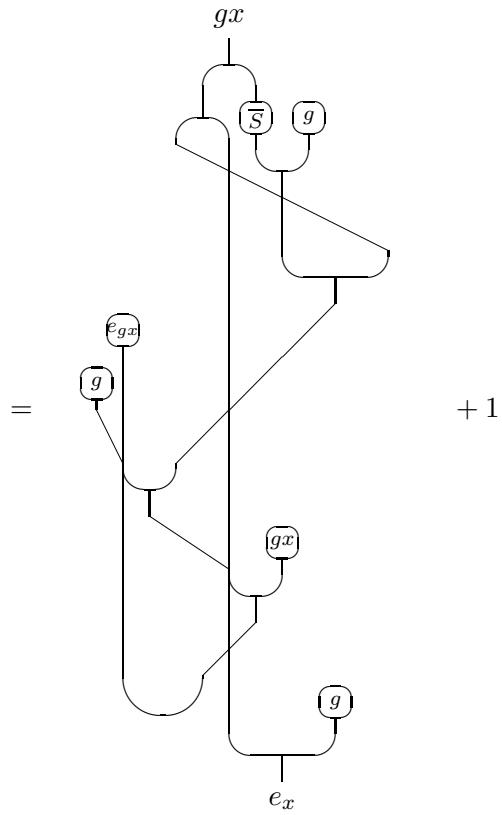




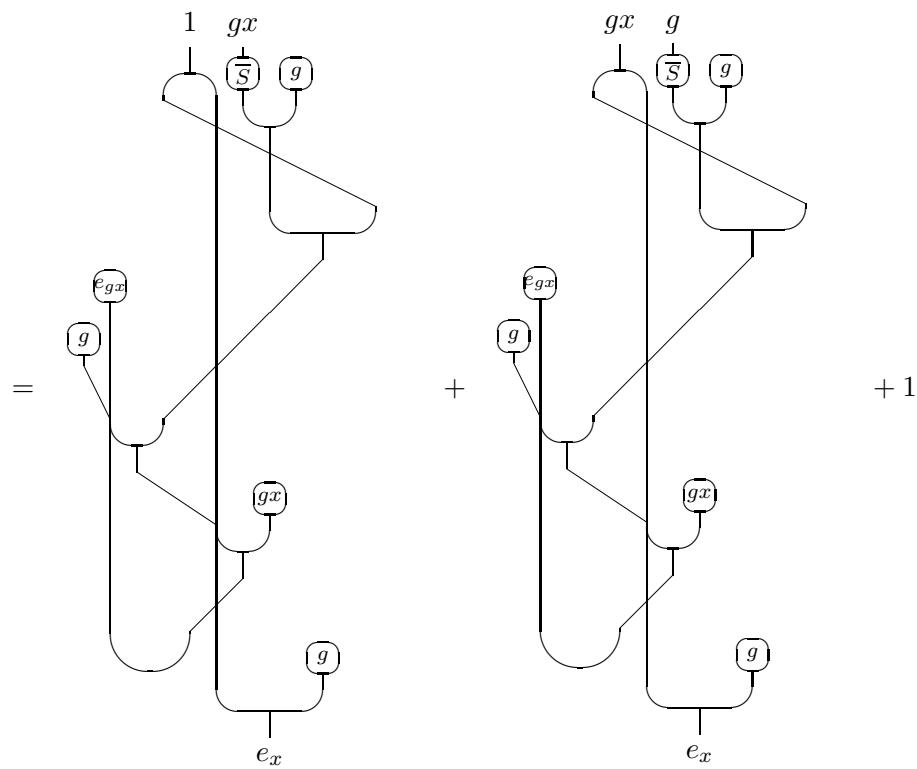
Let u and v denote the first term and the second term, respectively.

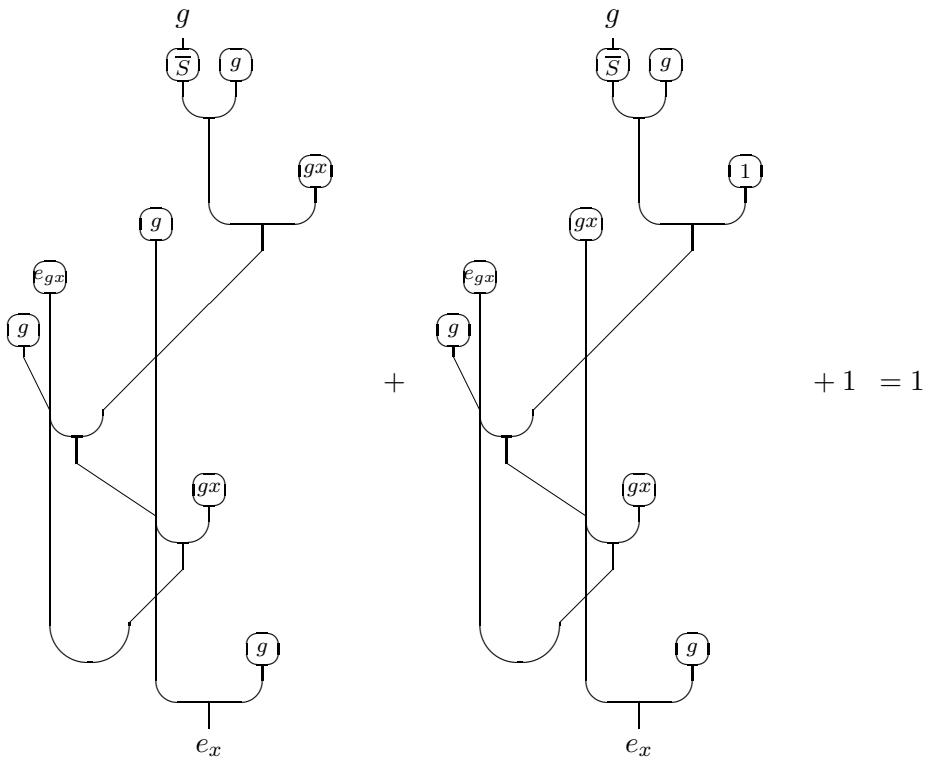




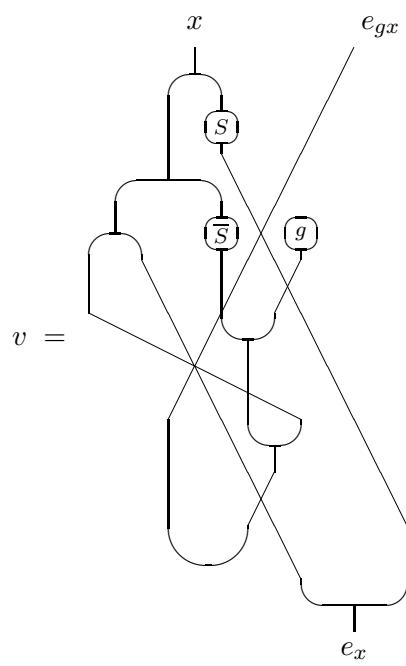


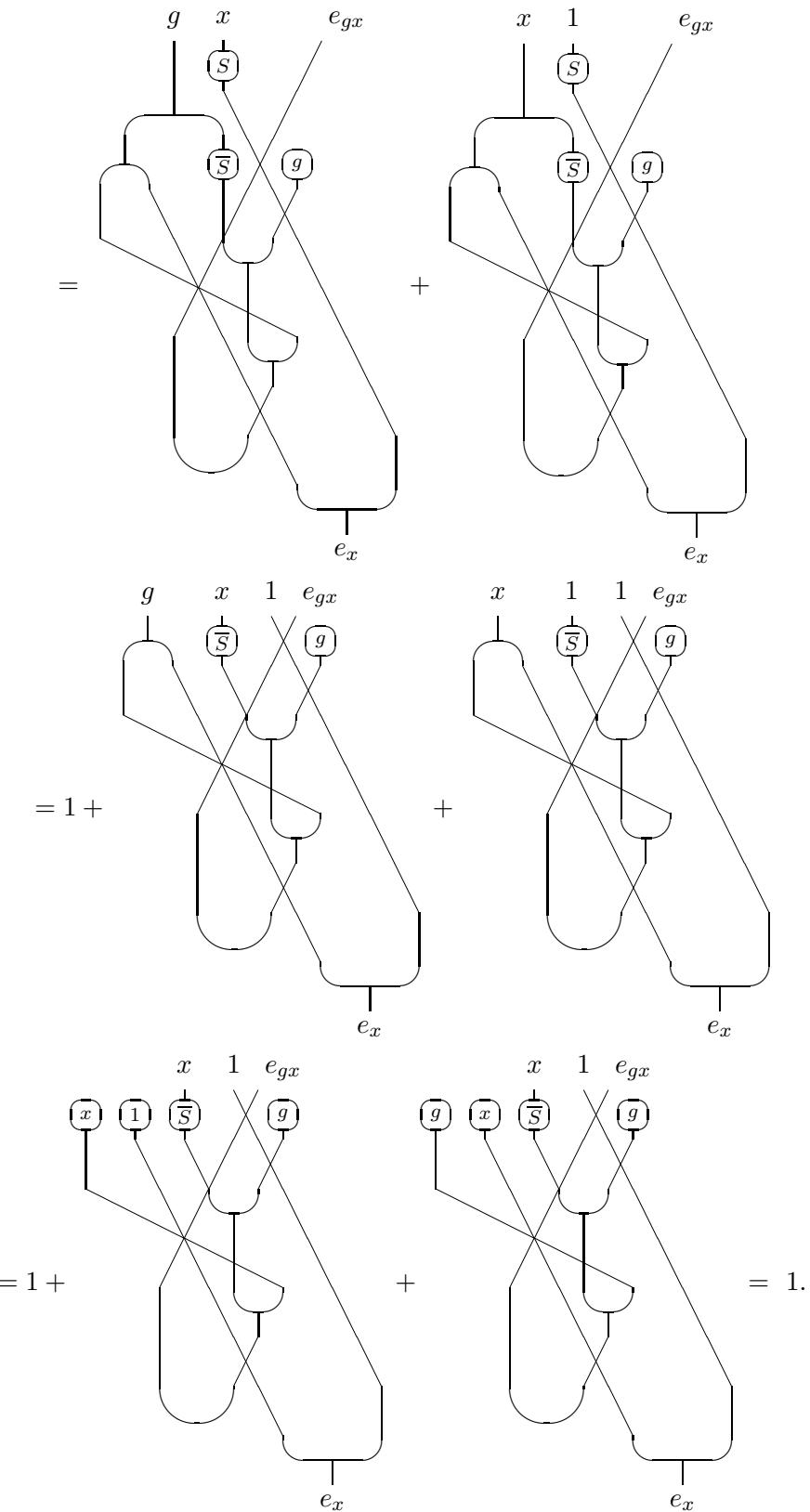
$+ 1$





and





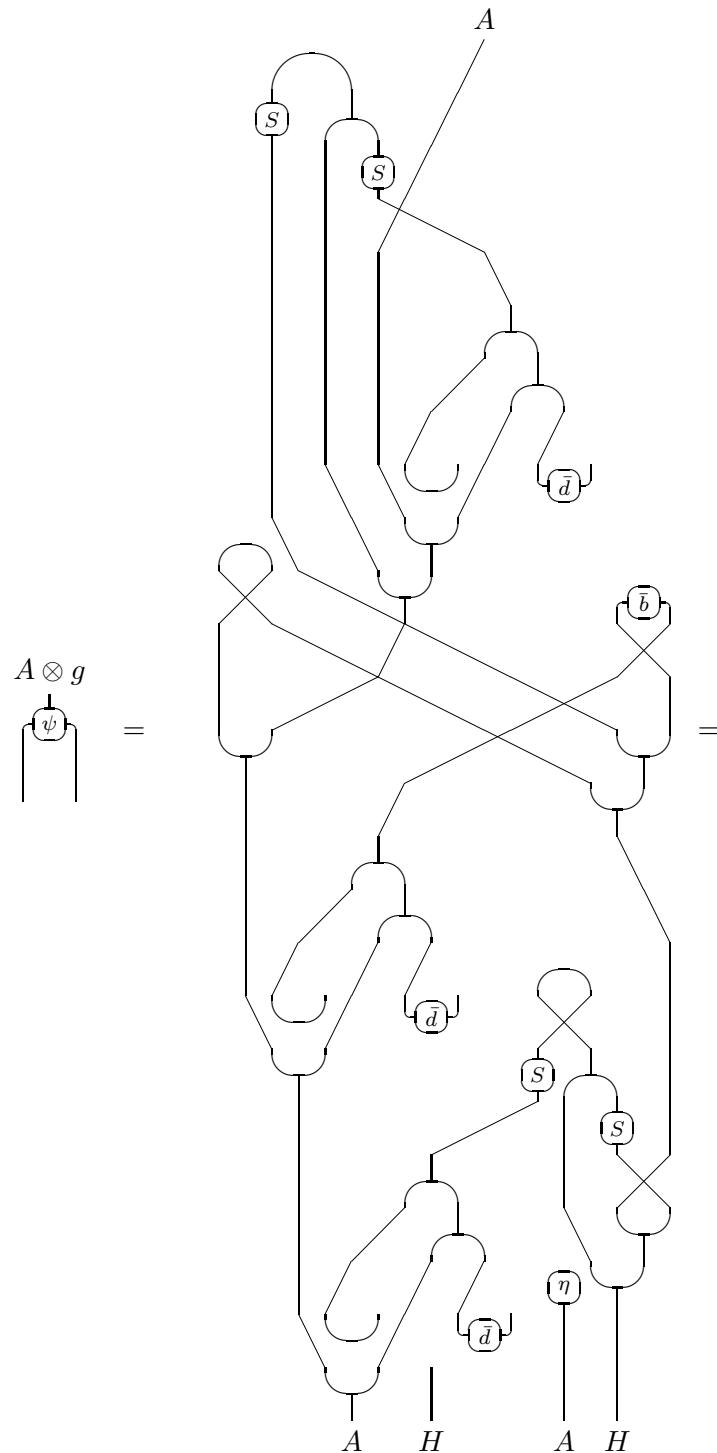
$$\text{Thus } \begin{array}{c} e_{gx} \otimes \eta \\ \phi \\ \hline x \otimes \epsilon g x \otimes e_x \end{array} = 2, \quad \text{but} \quad \begin{array}{c} \eta_D & e_{gx} \otimes \eta \\ \hline x \otimes \epsilon & g x \otimes e_x \end{array} = 0.$$

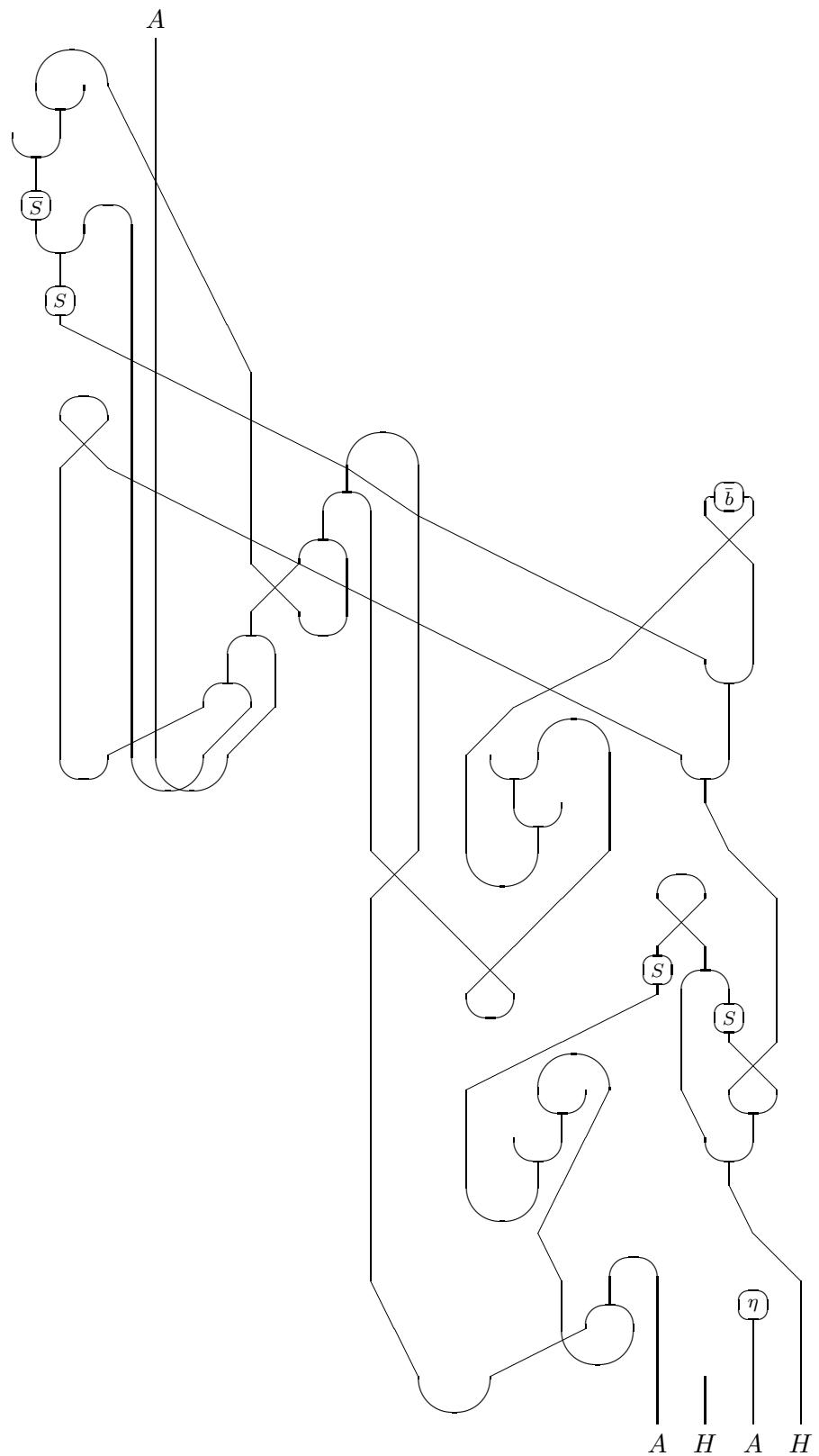
Consequently, ϕ is not trivial.

$$\begin{array}{c} \frac{\underline{D}}{\psi} = \\ \begin{array}{c} D \\ \hline D \quad D \end{array} \\ = \\ \begin{array}{c} S \\ \hline [b] \\ \hline S \\ \hline [b] \\ \hline [b] \\ \hline S \\ \hline ad \\ \hline D \quad D \end{array} \end{array}$$

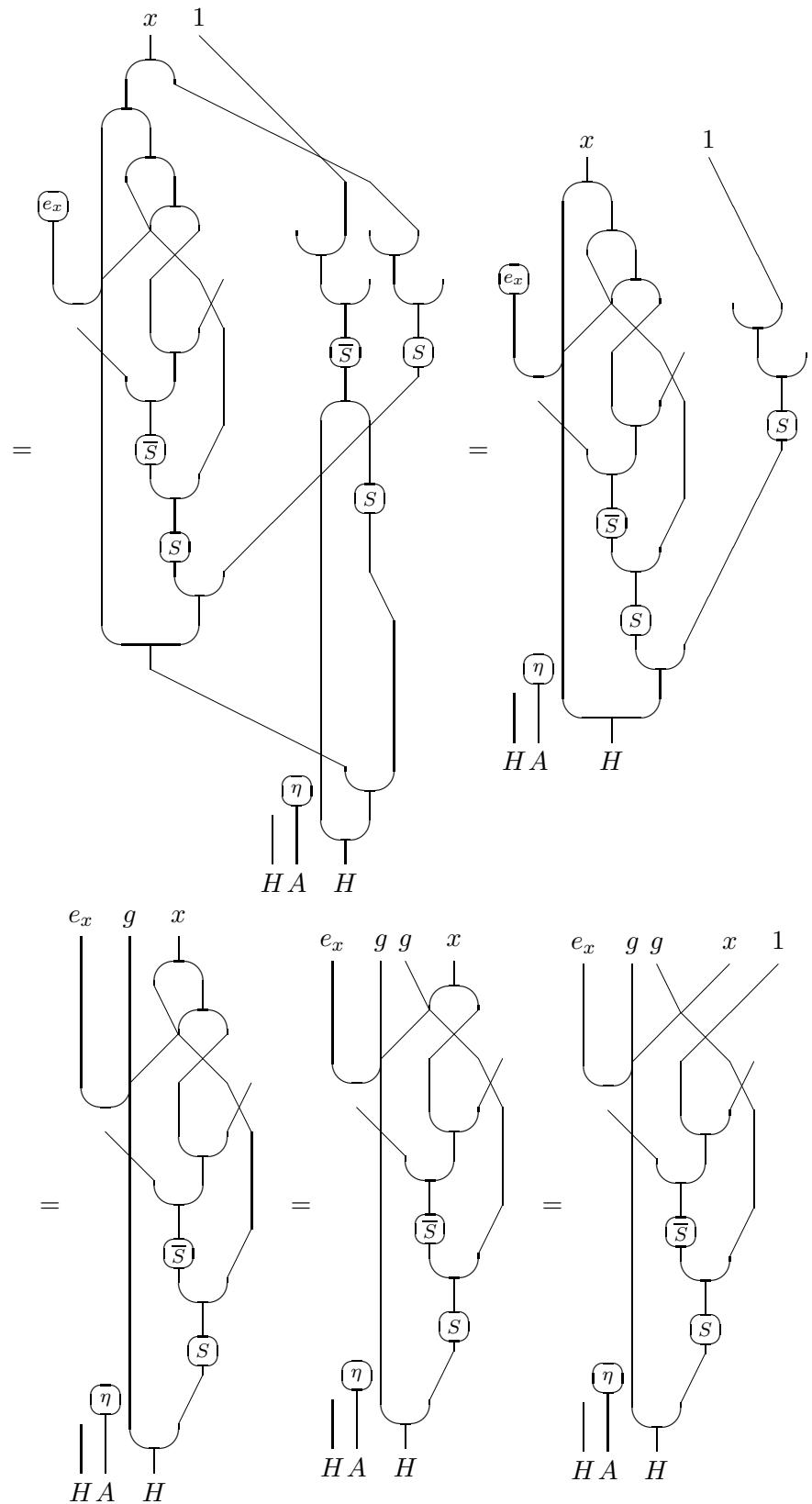
$$= \begin{array}{c} S \\ \hline [b] \\ \hline S \\ \hline [b] \\ \hline [b] \\ \hline S \\ \hline ad \\ \hline A \quad H \\ \hline \lambda \\ \hline S \\ \hline \lambda \\ \hline \lambda \\ \hline \eta \\ \hline A \quad H \end{array}.$$

For convenience, we omit g in the following diagrams:





and



$$= \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad , \text{ but} \quad \begin{array}{c} e_g \otimes g & \eta_D \\ | & | \\ x \otimes id & D \end{array}$$

Thus ψ is not trivial. \square

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